SAARLAND UNIVERSITY
Department of Mathematics
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Mathematics for Computer Scientists 1, WS 2018/19
Sheet 8

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1. Prove using the prime decomposition theorem that $\sqrt{n}$ is irrational for each $n \in \mathbb{N}$ with $n \neq m^2$ for some $m \in \mathbb{N}$.

2. a) Show that the sum of a irrational number and a rational number is irrational.

b) Show that the product of an irrational number and a non-zero rational number is irrational.

c) Give a counterexample to the assertion that the sum and product of two irrational numbers is rational.

d) Give a counterexample to the assertion that the sum and product of two irrational numbers is irrational.

3. (a) Let X be a non-empty set of real numbers which is bounded above. Prove that there is a sequence $\{x_n\}$ of numbers in X which converges to $\sup X$. (The sequence $\{x_n\}$ is called a *maximising sequence* for X.)

[Hint: Use the penultimate lemma in Section 3.3 with $\varepsilon = \frac{1}{n}$.]

Formulate a corresponding result for minimising sequences.

(b) Let r be an arbitrary real number. Prove that there exists a sequence $\{q_n\}$ of rational numbers which converges to r.

[Hint: Use the final remark in Section 3.4 with $\varepsilon = \frac{1}{n}$.]

4. Prove that

$$\lim_{n\to\infty} \lim_{k\to\infty} (\cos n!\pi x)^{2k} = \begin{cases} 1 \text{ if } x \text{ is rational,} \\ \\ 0 \text{ if } x \text{ is irrational.} \end{cases}$$

[Hint: If x is rational, then $n!\pi x$ is an integer multiple of π for large values of n. On the other hand, if x is irrational, then $n!\pi x$ is never an integer multiple of π.]