SAARLAND UNIVERSITY
Department of Mathematics
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Mathematics for Computer Scientists 1, WS 2018/19
Sheet 9

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1. Prove the following statements.

(a) $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$

(b) $\lim\limits_{n\to\infty} \dfrac{n+3}{n^3+4} = 0$

(c) $\lim\limits_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[8]{n^2} = 0$

(d) $\lim\limits_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = 0$

(e) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$

[Hint: $\dfrac{2}{n}$, $\dfrac{3}{n}$, $\ldots \dfrac{n-1}{n} \leq 1$.]

(f) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

[Hint: For each $\varepsilon > 0$ one has that $\dfrac{n}{(1+\varepsilon)^n} \to 0$ as $n \to \infty$, and in particular there exists $N \in \mathbb{N}$ such that $\dfrac{n}{(1+\varepsilon)^n} < 1$ for all $n > N$.]

(g) $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$ für $a > 0$

[Hint: There exists $N \in \mathbb{N}$ with $\dfrac{1}{n} < a < n$ for all $n > N$.]

(h) $\lim\limits_{n\to\infty} \sqrt[n]{a^n+b^n} = a$ für $a > b > 0$

[Hint: $\sqrt[n]{a^n+b^n} = a\sqrt[n]{1+\left(\frac{b}{a}\right)^n}$ and $1 < 1 + \left(\frac{b}{a}\right)^n < 2$.]

2. The sequence $\{x_n\}$ is determined by the recursive scheme

$$x_1 = 2, \qquad x_{n+1} = 1 + \frac{6}{x_n}, \quad n = 1, 2, 3, \ldots.$$

Show that

$$\text{(i) } x_n \in [2,4] \;\Rightarrow\; x_{n+1} \in [2,4], \qquad \text{(ii) } x_{n+2} = 7 - \frac{36}{x_n+6},$$

$$\text{(iii) } x_{n+2} \geq x_n \;\Rightarrow\; x_{n+3} \leq x_{n+1}, \qquad \text{(iv) } x_{n+2} \leq x_n \;\Rightarrow\; x_{n+3} \geq x_{n+1}$$

für $n = 1, 2, 3, \ldots$.

Prove that the sequences x_1, x_3, x_5, $\ldots$ and x_2, x_4, x_6, $\ldots$ converge and determine their limits. Deduce that $\{x_n\}$ converges and determine its limit.

3. Consider the sequence $\{a_n\}$, where $a_n = (1 + \frac{1}{n})^n$.

(a) Show that

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n}, \qquad n \in \mathbb{N}.$$

(b) Use Bernoulli's inequality

$$(1+x)^k \geq 1 + kx, \qquad x \geq -1,\ k \in \mathbb{N},$$

to prove the estimate

$$\frac{a_{n+1}}{a_n} \geq 1, \qquad n \in \mathbb{N}.$$

(c) Use the binomial expansion

$$(1+x)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j, \qquad |x| < 1,\ n \in \mathbb{N},$$

to prove the estimate

$$a_n \leq \sum_{j=0}^{n} \frac{1}{j!}, \qquad n \in \mathbb{N}.$$

[Hint:

$$\frac{n!}{(n-j)!}\frac{1}{n^j} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \ldots \cdot \frac{n-j+1}{n}$$

for $n \in \mathbb{N}$ and $j = 0, 1, \ldots, n$.]

(d) Prove that

$$a_n \leq 3, \qquad n \in \mathbb{N}.$$

[Hint: $2^{j-1} \leq j!$ for all $j \in \mathbb{N}$.]

(e) Deduce that $\{a_n\}$ converges to a real number in the interval $(2, 3)$. (This number is *Euler's number* e.)