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Department of Mathematics
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Mathematics for Computer Scientists 2, SS 2018
Sheet 1

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1. Use a mid-point Riemann sum and an equidistant partition of $[0,5]$ with 50 subintervals to approximate the value of the integral

$$\int_0^5 \sin x \cos(x - \tfrac{\pi}{5})\,\mathrm{d}x.$$

(Given your answer correct to four decimal places.)

2. a) Let $f : [a,b] \to \mathbb{R}$ be given by $f(x) = c$, where c is a constant, and let Z be an arbitrary partition of $[a,b]$.

Show that the upper Riemann sum $O_f(Z)$ and the lower Riemann sum $U_f(Z)$ are both equal to $c(b-a)$. Deduce that f is integrable over $[a,b]$ with $\int_a^b f = c(b-a)$.

b) A function $s : [a,b] \to \mathbb{R}$ is called a *step function* if there exists a partition $Z = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that f is constant on each subinterval (x_{j-1}, x_j), $j = 1, \ldots, n$.

Show that every step function $f : [a,b] \to \mathbb{R}$ is integrable and calculate its integral.

3. Let f, $g : [a,b] \to \mathbb{R}$ be integrable functions. Prove the following assertions.

(i) It follows from $f(x) \geq 0$ for all $x \in [a,b]$ that $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$.

[Hint: Which sign do the upper and lower Riemann sums have?]

(ii) It follows from $f(x) \leq g(x)$ for all $x \in [a,b]$ that $\displaystyle\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$.

(iii) It follows from $m \leq f(x) \leq M$ for all $x \in [a,b]$ that $m(b-a) \leq \displaystyle\int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$.

4. a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Prove the following assertion using the inequality

$$O_{|f|}(Z) - U_{|f|}(Z) \leq O_f(Z) - U_f(Z),$$

which is valid for every partition Z of $[a,b]$, and the Riemann integrability criterion:

If f is integrable over $[a,b]$, then $|f|$ is also integrable over $[a,b]$.

b) Suppose that f is integrable. Prove that $\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$

c) Find a nonintegrable function $f : [a,b] \to \mathbb{R}$ with the property that $|f|$ is integrable.

d) The *positive part* and *negative part* of a function $f : [a,b] \to \mathbb{R}$ are the functions $f_\pm : [a,b] \to \mathbb{R}$ given by the formulae

$$f_+(x) = \begin{cases} f(x), & f(x) \geq 0, \\ 0, & f(x) < 0, \end{cases} \qquad f_-(x) = \begin{cases} 0, & f(x) > 0, \\ -f(x), & f(x) \leq 0. \end{cases}$$

Suppose that f is integrable. Prove that $f_\pm$ are also integrable over $[a,b]$.