**SAARLAND UNIVERSITY**
**Department of Mathematics**
**Prof. Dr. Mark Groves**
**MSc Jens Horn**

**Mathematics for Computer Scientists 2, SS 2018**
**Sheet 7**

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**1.** Let

$$M_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ -1 & 1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}, \qquad M_5 = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix}.$$

Compute the defined products $M_iM_j$ for $i, j = 1, \ldots, 6$.

**2.** (a) Let $\mathcal{B}_3, \mathcal{B}_2$ be the usual bases for $\mathbb{R}^3$ and $\mathbb{R}^2$,

$$\mathcal{B}_3' = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{B}_2' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

be further bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, and $T : \mathbb{R}^3 \to \mathbb{R}^2$, $S : \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformations with

$$M_{\mathcal{B}_2}^{\mathcal{B}_3}(T) = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{pmatrix}, \qquad M_{\mathcal{B}_3'}^{\mathcal{B}_2'}(S) = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Compute $M_{\mathcal{B}_2'}^{\mathcal{B}_3'}(T)$ and $M_{\mathcal{B}_3}^{\mathcal{B}_2}(S)$.

[Hint: $B_{\mathcal{B}_3}^{\mathcal{B}_3'}$ and $B_{\mathcal{B}_2'}^{\mathcal{B}_2}$ are computed in the example on pages 83–84 in the lecture notes.]

(b) Let $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the usual basis for $\mathbb{R}^{2\times 2}$,

$$\mathcal{B}' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be a further basis for $\mathbb{R}^{2\times 2}$ and $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation with

$$M_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Compute $M_{\mathcal{B}'}^{\mathcal{B}'}(T)$.

**3.** Let $\mathcal{A} = \{x^2, x^2+1, x-1\}$ and $\mathcal{B} = \{x, 1-x, x^2\}$ be bases for $\mathcal{P}_2(\mathbb{R})$.

(a) Compute $M_{\mathcal{B}}^{\mathcal{A}}$ and $M_{\mathcal{A}}^{\mathcal{B}}$.

(b) Find a basis $\mathcal{C}$ for $\mathcal{P}_2(\mathbb{R})$ such that $M_{\mathcal{B}}^{\mathcal{C}} = M_{\mathcal{A}}^{\mathcal{B}}$.

**4.** Let $K$ be a field, $A \in K^{m\times n}$ and $B \in K^{n\times p}$.

(a) Prove that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.

[Hint: Using the remark on page 86 of the lecture notes, one can write down formulae for $\mathbf{c}_j^{AB}$ and $\mathbf{r}_j^{B^{\mathrm{T}}A^{\mathrm{T}}}$.]

(b) Show that

$$\text{column rank}\, AB \leq \text{column rank}\, A \qquad \text{row rank}\, AB \leq \text{row rank}\, B$$

and deduce that

$$\text{rank}\, AB \leq \min(\text{rank}\, A, \text{rank}\, B).$$

[Hint: Read the remark on page 86 of the lecture notes.]