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Mathematics for Computer Scientists 2, SS 2018
Examination preparation

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1. Compute the following indefinite integrals.

(i) $\displaystyle\int \frac{x+3}{x+5}\,\mathrm{d}x$, (v) $\displaystyle\int \frac{\sin^3 x}{\cos^5 x}\,\mathrm{d}x$, (ix) $\displaystyle\int \mathrm{e}^x \sin x\,\mathrm{d}x$,

(ii) $\displaystyle\int \frac{x^2+2x+7}{x+5}\,\mathrm{d}x$, (vi) $\displaystyle\int (5x^4+4x^3)\mathrm{e}^{x^5+x^4}\,\mathrm{d}x$, (x) $\displaystyle\int x^{-2}\sin x^{-1}\,\mathrm{d}x$,

(iii) $\displaystyle\int \frac{x}{x^4+1}\,\mathrm{d}x$, (vii) $\displaystyle\int x^2 \log x\,\mathrm{d}x$, (xi) $\displaystyle\int x^2 \mathrm{e}^{2x}\,\mathrm{d}x$,

(iv) $\displaystyle\int \frac{\mathrm{e}^{3x}}{\mathrm{e}^{3x}+5}\,\mathrm{d}x$, (viii) $\displaystyle\int \frac{\log\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$, (xii) $\displaystyle\int x^2 \sin 3x\,\mathrm{d}x$.

2. Consider the subspace

$$W = \{(x,y,z,t) \in \mathbb{R}^4) : x = 0,\ y+2z+4t = 0\}$$

of $\mathbb{R}^4$. Find a basis for W and hence determine $\dim W$. Extend this basis to a basis for $\mathbb{R}^4$ and hence find a complement of W in $\mathbb{R}^4$.

3. Show that

$$\{x^3+x^2, x^3+x\}$$

is a basis for the subspace

$$W = \{p \in \mathcal{P}_3(\mathbb{R}) : p(-1) = p(0) = 0\}$$

of $\mathcal{P}_3(\mathbb{R})$ and hence determine $\dim W$. Extend this basis to a basis for $\mathcal{P}_3(\mathbb{R})$ and hence find a complement of W in $\mathcal{P}_3(\mathbb{R})$.

4. Let $\mathcal{B}$ be the usual basis for $\mathbb{R}^4$,

$$\mathcal{B}' = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\}$$

be a further basis for $\mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^4$, $S : \mathbb{R}^4 \to \mathbb{R}^3$ be linear transformations with

$$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1+x_3\\2x_1-x_2\\x_2+2x_3\\-x_3-x_1\end{pmatrix}, \qquad S\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}x_1+x_3+x_4\\2x_1-x_2\\x_2+2x_3-x_4\end{pmatrix}.$$

Compute $M_{\mathcal{B}}^{\mathcal{B}'}(T)$ and $M_{\mathcal{B}'}^{\mathcal{B}}(S)$.

5. Consider the subspace

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}\}$$

of $\mathbb{R}^{3\times 3}$.

(i) Let $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ be the linear transformation

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 2a_{12} & 3a_{13} \\ 4a_{21} & 5a_{22} & 4a_{23} \\ 7a_{31} & a_{32} & 9a_{33} \end{pmatrix}.$$

Find the matrix of T with respect to the basis $\mathcal{B}$.

(ii) Let $M \in \mathbb{R}^{3\times 3}$ be the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 9 \end{pmatrix}$$

and $T_M : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ be the linear transformation with

$$T_M(A) = MA.$$

Find the matrix of T_M with respect to the basis $\mathcal{B}$.

6. Let $\mathcal{B}$ be the usual basis for $\mathbb{R}^4$,

$$\mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

be a further basis for $\mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation with

$$M_{\mathcal{B}'}^{\mathcal{B}'}(T) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Compute $M_{\mathcal{B}}^{\mathcal{B}}(T)$.

7. Determine for which values of λ the real matrix

$$A_\lambda = \begin{pmatrix} \lambda & 1 & \lambda \\ -1 & 1 & -1 \\ 1 & 1 & \lambda \end{pmatrix}$$

is invertible, and compute the inverse matrix A_λ^{-1} for these values of λ.

8. Determine for which values of a, b the real matrix

$$A_{a,b} = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 4 & 4 \\ 2 & 7 & a & b \end{pmatrix}$$

is invertible, and compute the inverse matrix $A_{a,b}^{-1}$ for these values of a and b.

9. Find all real nontrivial solutions of the equations

$$\begin{aligned} x + 2y + 2z + 2t &= 2, \\ 2x + 3t &= 0, \\ 3x + 10y + 12z + 6t &= 6, \\ 4x + 2y + 3z + 6t &= 0 \end{aligned}$$

and

$$\begin{aligned} 3a + b - c &= 3, \\ a + 2b &= 2, \\ -2a - 3b + c &= -1. \end{aligned}$$

10. Solve the real systems of linear equations

(a) $$\begin{aligned} x + y + z &= 2, \\ x - y - z &= 0, \\ \lambda y + z &= 1, \end{aligned}$$

(b) $$\begin{aligned} a + 2b + 3c &= \mu, \\ 2a + 1b + 3c &= \mu, \\ 2b + 2c &= 2. \end{aligned}$$

11. Compute the determinant of the matrices

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 3 & 2 & 0 & 2 \\ 0 & 1 & -1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 \\ 1 & -1 & 2 & 1 & -1 \\ 0 & 2 & 0 & 5 & 6 \\ -1 & 1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

12. Suppose that $n \in \mathbb{N}$, $b_i, c_i \in \mathbb{R}$ for $i = 1, \dots, n$ and

$$G_n = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & b_n \\ 0 & 1 & \ddots & & \vdots & b_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & 1 & b_2 \\ c_n & c_{n-1} & \dots & \dots & c_2 & 1 \end{pmatrix}$$

for $n \in \mathbb{N}$. Show that

$$\det(G_n) = 1 - b_2c_2 - \cdots - b_nc_n$$

for every $n \geq 2$.

13. Compute the eigenvalues and eigenspaces of the complex matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & -1 & 3 \\ -2 & 2 & -3 \\ 2 & -1 & 4 \end{pmatrix}, \quad \begin{pmatrix} -2 & -2 & 1 \\ 2 & 3 & -2 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 2 & -4 \\ 2 & 3 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

14. Which of the matrices

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

are diagonalisable over $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$?

15. Let

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

(i) Compute the eigenvalues and eigenspaces of the matrices A_1, A_2, A_3.

(ii) Find a real, invertible matrix P_i such that $P_i^{-1} A_i P_i$ is diagonal for $i = 1, 2, 3$.

(iii) Find the Sylvester normal form of A_i for $i = 1, 2, 3$.

16. Show that

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

is a basis for $\mathbb{R}^4$. Apply the Gram-Schmidt procedure to S to find an orthonormal basis for $\mathbb{R}^4$.

17. Find an orthonormal basis for the subspace

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathrm{i} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \\ \mathrm{i} \end{pmatrix} \right\rangle$$

of $\mathbb{C}^4$ and extend it to an orthonormal basis for $\mathbb{C}^4$.

18. Consider the subspace $V = \langle v_1, v_2, v_3, v_4 \rangle$ of $\mathcal{P}(\mathbb{R})$, where

$$v_1(x) = 1, \qquad v_2(x) = (2x-1)\,, \qquad v_3(x) = x\,(3x-2)\,, \qquad v_4(x) = x^2\,(4x-3)\,.$$

Apply the Gram-Schmidt procedure to find an orthonormal basis for V with respect to the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x, \qquad p, q \in \mathcal{P}(\mathbb{R}).$$

19. Show that the matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

lies in $\mathrm{SO}(3)$. Determine the axis and angle of the rotation represented by A.

20. Show that the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

lies in $\mathrm{SO}(3)$. Determine the axis and angle of the rotation represented by A.

21. Sketch the conic sections with equations

(i) $2y^2 - 8x = 0$,

(ii) $16x^2 - 9y^2 = 144$,

(iii) $4x^2 - 4x + 1 = 0$,

(iv) $xy - 4x + 2y - 4 = 0$,

(v) $x^2 - 2xy + 2y^2 + 2x - 6y + 1 = 0$,

(vi) $x^2 - 2xy + y^2 + 4x + 4y - 20 = 0$,

(vii) $4x^2 - 4xy + 3y^2 + 8x - 4y + 3 = 0$.