SAARLAND UNIVERSITY
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Mathematics for Computer Scientists 2, SS 2018
Sheet 6

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1. Which of the following transformations are linear?

(i) $\mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + 2y$

(ii) $\mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + y^2$

(iii) $\mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto xy$

(iv) $\mathbb{C} \to \mathbb{C}, \quad z \mapsto \overline{z}$

(v) $\mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 1 \\ y - 1 \end{pmatrix}$

(vi) $\mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ x + 2y \end{pmatrix}$

(vii) $\mathcal{P}_n(\mathbb{R}) \to \mathbb{R}, \quad p(x) \mapsto p(1)$

(viii) $\mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_{n+2}(\mathbb{R}), \quad p(x) \mapsto x^2 p(x)$

2. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by the formula

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x + 2y - z \\ 2x + y + z \end{pmatrix}.$$

Find the matrix of T with respect to the usual basis for $\mathbb{R}^3$.

(b) Let $n \in \mathbb{N}$ and $T : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ be the linear transformation defined by

$$(T(p))(x) = p(x + 1).$$

Find the matrix of T with respect to the usual basis for $\mathcal{P}_n(\mathbb{R})$.

(c) Let $T : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ be the linear transformation defined by the formula

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2b \\ 3c & 4d \end{pmatrix}.$$

Find the matrix of T with respect to the usual basis for $\mathbb{R}^{2 \times 2}$.

3. The matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the usual basis for $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Find the matrix of T with respect to the basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

for $\mathbb{R}^3$.

[Hint: Use the matrix A to find a formula for $T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.]

4. (a) Let U, V, W be vector spaces over a field K and $S : U \to V$, $T : V \to W$ be isomorphisms. Prove that $S^{-1} : V \to U$ and $T \circ S : U \to W$ are also isomorphisms.

(b) Let M be the set of all vector spaces over a field K. Prove that the formula

$$V \sim W \qquad \Leftrightarrow \qquad V \cong W$$

defines an equivalence relation on M.

(c) Let V and W be two finite-dimensional, isomorphic vector spaces over a field K. Prove that $\dim V = \dim W$.

[Hint: Let $\{e_1, \ldots, e_n\}$ be a basis for V and $T : V \to W$ be an isomorphism. Prove that $\{T(e_1), \ldots, T(e_n)\}$ is a basis for W.]