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Department of Mathematics
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Mathematics for Computer Scientists 2, SS 2018
Sheet 4

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1. Show that

$$T = \left\{ \begin{pmatrix} 1 \\ \mathrm{i} \\ 1+\mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \mathrm{i} \end{pmatrix} \right\}$$

is a linearly independent subset of $\mathbb{C}^3$ and

$$S = \left\{ \begin{pmatrix} \mathrm{i} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathrm{i} \\ 1 \end{pmatrix} \right\}$$

is a spanning set for $\mathbb{C}^3$. Use the algorithm in the Steinitz exchange theorem to replace two elements of S with elements of T.

2. (a) Show that the set $\mathbb{C}^2$ is a complex vector space with respect to the vector addition and scalar multiplication defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + y_1 + 1 \\ x_2 + y_2 + 1 \end{pmatrix}, \qquad \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \alpha x_1 + \alpha - 1 \\ \alpha x_2 + \alpha - 1 \end{pmatrix}.$$

Are the vectors $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ und $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$ linearly independent in this vector space?

(b) Let X be an arbitrary set. Show that the power set $\mathcal{P}(X)$ is a vector space over the trivial field $\{0, 1\}$ with respect to the vector addition and scalar multiplication defined by

$$Y_1 + Y_2 := Y_1 \Delta Y_2$$

and

$$0Y := \emptyset, \qquad 1Y := Y.$$

[Note: The power set $\mathcal{P}(X)$ is the set of all subsets of X. The *symmetric difference* of two sets A and B is $A \Delta B := (A \cup B) \setminus (A \cap B)$.]

3. Let V be be a vector space and $v_1, v_2, \ldots, v_n \in V$. Prove the following assertions.

(a) If $\langle v_1, \ldots, v_n \rangle = V$, then $\langle v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n \rangle = V$.

(b) If $v_1, v_2, \ldots, v_n$ are linearly independent, then $v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n$ are also linearly independent.

4. Let X be a nonempty set and $+$ be an associative binary operation on X with the following properties.

(i) The element $0 \in X$ satisfies $0 + x = x$ for all $x \in X$.

(ii) For each $x \in X$ there is an element $-x$ with $-x + x = 0$.

Prove that $x + 0 = x$ for all $x \in X$ and $x + (-x) = 0$. Prove further that 0 is the only element in X with the property (i) and $-x$ is the only element in X such that $-x + x = 0$.

What are the implications of this result for the vector space axioms (V1)–(V4)?