SAARLAND UNIVERSITY
Department of Mathematics
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Mathematics for Computer Scientists 2, SS 2018
Sheet 10

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1. Let

$$A_n = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 4 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & (n-1)^2 & 1 \end{pmatrix}, \qquad n = 1, 2, 3, \ldots.$$

Compute $\det(A_1)$ and $\det(A_2)$, find a formula for $\det(A_n)$ for $n \geq 3$ as a function of $\det(A_{n-1})$ and $\det(A_{n-2})$ and prove by strong induction that

$$\det(A_n) = n!, \qquad n = 1, 2, 3, \ldots.$$

2. Compute the eigenvalues and eigenspaces of the complex matrices

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 2 & \mathrm{i} & 1+2\mathrm{i} \\ -\mathrm{i} & 0 & -\mathrm{i} \\ 1-2\mathrm{i} & \mathrm{i} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

3. Let K be a field. The *trace* of a matrix in $K^{n\times n}$ is the sum of its diagonal entries.

(a) Prove that $\operatorname{tr} AB = \operatorname{tr} BA$ for all A, $B \in K^{n\times n}$, and that two similar matrices in $K^{n\times n}$ have the same trace. How would you define the trace of a linear transformation $T : V \to V$ for an n-dimensional vector space V over K?

(b) Let $A \in K^{n\times n}$ and c be its characterisic polynomial. Show that

(i) the coefficient of λ^n in c is $(-1)^n$,

(ii) the coefficient of λ^{n-1} in c is $(-1)^{n-1} \operatorname{tr} A$,

(iii) the coefficient of λ^0 in c is $\det A$.

[Hint: Study the proof that c is a polynomial of degree n.]